

# STABILITY OF SOLUTIONS TO NONLINEAR DIFFUSION EQUATIONS

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**ABSTRACT.** We prove stability results for nonlinear diffusion equations of the porous medium and fast diffusion types with respect to the nonlinearity power  $m$ : solutions with fixed data converge in a suitable sense to the solution of the limit problem with the same data as  $m$  varies. Our arguments are elementary and based on a general principle. We use neither regularity theory nor nonlinear semigroups, and our approach applies to e.g. Dirichlet problems in bounded domains and Cauchy problems on the whole space.

## 1. INTRODUCTION

We study the stability of positive solutions to the parabolic equation

$$(1.1) \quad \partial_t u - \Delta u^m = 0$$

with respect to perturbations in the nonlinearity power  $m$ . The main issue we address is whether solutions with fixed boundary and initial data converge in some sense to the solution of the limit problem as  $m$  varies. This kind of stability questions are not only a matter of merely mathematical interest; in applications, parameters like  $m$  are often known only approximately, for instance from experiments. Thus it is natural to ask whether solutions are sensitive to small variations in such parameters or not.

The equation (1.1) is an important prototype of a nonlinear diffusion equation. For  $m > 1$ , this is called the porous medium equation (PME), and  $m < 1$ , the fast diffusion equation (FDE). The PME is degenerate, the diffusion being slow when  $u$  is small. The FDE is singular, and the opposite happens: the diffusion is fast when  $u$  is small. We do not exclude the case  $m = 1$ , when we have the ordinary heat equation.

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However, we do restrict our attention to the supercritical range:

$$m > m_c, \quad \text{where} \quad m_c = (n - 2)_+/n.$$

For the basic theory of the porous medium and fast diffusion equations, we refer to the monographs [6, 26, 27] and the references therein.

It turns out that stability with respect to  $m$  for (1.1) can be established in a relatively elementary manner. The principal reason for this is that weak solutions to (1.1) are defined in terms of the function  $u^m$  instead of  $u$ . This means that  $u^m$  and its gradient are always  $L^2$  functions, even if  $m$  varies. The situation is markedly different for the  $p$ -parabolic equation,

$$(1.2) \quad \partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

where changing the exponent  $p$  also changes the space to which weak solutions belong. Dealing with this in the case of (1.2) is quite delicate, and one needs the fact that the gradient satisfies a reverse Hölder inequality. The equation (1.1) has the advantage that we may work in a fixed space. For this reason, our results do not rely on the more sophisticated tools of regularity theory. More specifically, we do not need Hölder continuity, Harnack's inequality, or reverse Hölder inequalities for the gradient.

The starting point of our argument is a compactness property of weak solutions: locally uniformly bounded sequences of weak solutions contain pointwise almost everywhere convergent subsequences. With the compactness result in hand, a stability result with convergence in an  $L^p$  space for a particular problem follows by verifying the local uniform boundedness and that the correct initial or boundary values are attained. Only some fairly simple estimates are needed for the second step. We carry out this step in detail in two cases: for Dirichlet problems with nonzero boundary values on bounded domains, and for Cauchy problems on the whole space, with initial data a positive measure of finite mass. In the latter case, our stability result applies to the celebrated Barenblatt solutions.

The previous result closest to ours is that of Bénilan and Crandall [2]. They prove the stability of mild solutions to

$$(1.3) \quad \partial_t u - \Delta \varphi(u) = 0$$

with respect to  $\varphi$  using the theory of nonlinear semigroups. Here  $\varphi$  can be a maximal monotone graph; in the case of a power function, the assumption used in [2] reduces to  $m \geq m_c$ . Our approach is different from that of [2], as we do not employ the machinery of nonlinear semigroups. Explicit estimates covering the situation of [2] are established

in [4]. See also [21, 22] for the onedimensional situation. Stability results are also known for, e.g., equations similar to the  $p$ -Laplacian [17], the  $p$ -parabolic equation (1.2) [15], obstacle problems [16], and eigenvalue problems [18]. See also [3, 24] for the limit when  $m \rightarrow \infty$ , and [10, 12, 13] for  $m \rightarrow 0$ .

Generalizing our argument to other boundary and initial conditions is straightforward. For instance, Neumann boundary conditions and Cauchy problems with growing initial data can be handled similarly. Generalization to nonlinearities other than powers, as in (1.3), should also be possible, albeit less straightforward.

Our results include the case when the limiting problem is the heat equation. In this situation, we do not need the restrictions  $m \geq 1$  or  $m \leq 1$  on the approximating problems; both degenerate ( $m > 1$ ) and singular ( $m < 1$ ) problems are allowed. However, the restriction  $m > m_c$  seems essential, as a number of the tools we use are known to fail when  $0 < m \leq m_c$ . For instance, local weak solutions might no longer be locally bounded, and the  $L^1$ - $L^\infty$  smoothing effect for the Cauchy problem fails.

We also provide an alternative argument for the stability of Dirichlet problems in bounded domains in the case  $m \geq 1$ . Compared to the previous argument, the advantages of this approach are the fact that local boundedness is not needed, and that one can in addition estimate the difference of two solutions in terms of the difference of the respective nonlinearity powers. The disadvantages are the restriction  $m \geq 1$  and the fact that the proof seems less amenable to generalizations, since we employ strong monotonicity. See [21, 22] for similar estimates in the onedimensional case.

The paper is organized as follows. In Section 2, we recall the necessary background material, in particular the definition of weak solutions. Section 3 contains the proof of the compactness theorem for locally bounded sequences of weak solutions. The actual stability results are then established in Sections 4 and 5, for Dirichlet problems in the former and Cauchy problems in the latter. We finish by presenting the alternative proof for Dirichlet problems in Section 6.

## 2. WEAK SOLUTIONS

Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ , and let  $0 < t_1 < t_2 < T$ . We use the notation  $\Omega_T = \Omega \times (0, T)$  and  $U_{t_1, t_2} = U \times (t_1, t_2)$ , where  $U \subset \Omega$  is open. The parabolic boundary  $\partial_p U_{t_1, t_2}$  of a space-time cylinder  $U_{t_1, t_2}$  consists of the initial and lateral boundaries, i.e.

$$\partial_p U_{t_1, t_2} = (\overline{U} \times \{t_1\}) \cup (\partial U \times [t_1, t_2]).$$

The notation  $U_{t_1, t_2} \Subset \Omega_T$  means that the closure  $\overline{U_{t_1, t_2}}$  is compact and  $\overline{U_{t_1, t_2}} \subset \Omega_T$ .

We use  $H^1(\Omega)$  to denote the usual Sobolev space, the space of functions  $u$  in  $L^2(\Omega)$  such that the weak gradient exists and also belongs to  $L^2(\Omega)$ . The norm of  $H^1(\Omega)$  is

$$\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}.$$

The Sobolev space with zero boundary values, denoted by  $H_0^1(\Omega)$ , is the completion of  $C_0^\infty(\Omega)$  with respect to the norm of  $H^1(\Omega)$ . The dual of  $H_0^1(\Omega)$  is denoted by  $H^{-1}(\Omega)$ .

The parabolic Sobolev space  $L^2(0, T; H^1(\Omega))$  consists of measurable functions  $u : \Omega_T \rightarrow [-\infty, \infty]$  such that  $x \mapsto u(x, t)$  belongs to  $H^1(\Omega)$  for almost all  $t \in (0, T)$ , and

$$\int_{\Omega_T} |u|^2 + |\nabla u|^2 dx dt < \infty.$$

The definition of  $L^2(0, T; H_0^1(\Omega))$  is identical, apart from the requirement that  $x \mapsto u(x, t)$  belongs to  $H_0^1(\Omega)$ . We say that  $u$  belongs to  $L_{loc}^2(0, T; H_{loc}^1(\Omega))$  if  $u \in L^2(t_1, t_2; H^1(U))$  for all  $U_{t_1, t_2} \Subset \Omega_T$ .

We use the following Sobolev inequality. See [7, Proposition 3.1, p. 7] for the proof.

**Lemma 2.1.** *Let  $u$  be a function in  $L^2(0, T; H_0^1(\Omega))$ . Then we have*

$$(2.1) \quad \int_{\Omega_T} |u|^{2\kappa} dx dt \leq C \int_{\Omega_T} |\nabla u|^2 dx dt \left( \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} u^{1+1/m} dx \right)^{2/n},$$

where

$$(2.2) \quad \kappa = 1 + \frac{1}{n} + \frac{1}{mn}.$$

Solutions are defined in the weak sense in the parabolic Sobolev space.

**Definition 2.2.** Assume that  $m > m_c$ . A nonnegative function  $u : \Omega_T \rightarrow \mathbb{R}$  is a local weak solution of the equation

$$(2.3) \quad \frac{\partial u}{\partial t} - \Delta u^m = 0$$

in  $\Omega_T$ , if  $u^m \in L_{loc}^2(0, T; H_{loc}^1(\Omega))$  and

$$(2.4) \quad \int_{\Omega_T} -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi dx dt = 0$$

for all smooth test functions  $\varphi$  compactly supported in  $\Omega_T$ .

We will always assume that  $m > m_c$ , and consider only nonnegative solutions. We refer to the monographs [6, 26, 27] for the basic theory related to this type of equations, and numerous further references. In particular, weak solutions have a locally Hölder continuous representative, see [5, 8] or Chapter 7 of [27]; however, we do not use this fact.

An important example of a local weak solution in the sense of Definition 2.2 is provided by the celebrated Barenblatt solution [1, 28]. For  $m > 1$ , it is given by

$$(2.5) \quad \mathcal{B}_m(x, t) = \begin{cases} t^{-\lambda} \left( C - \frac{\lambda(m-1)}{2mn} \frac{|x|^2}{t^{2\lambda/n}} \right)_+^{1/(m-1)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where

$$\lambda = \frac{n}{n(m-1) + 2}.$$

For  $m_c < m < 1$ , it is convenient to write the formula as

$$(2.6) \quad \mathcal{B}_m(x, t) = \begin{cases} t^{-\lambda} \left( C + k \frac{|x|^2}{t^{2\lambda/n}} \right)^{-1/(1-m)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where

$$k = \frac{\lambda(1-m)}{2mn}.$$

Note that  $\lambda$  and  $k$  are strictly positive, since here  $m_c < m < 1$ .

Observe that for  $m > 1$ ,  $\mathcal{B}_m$  has compact support in space at each time instant, while for  $m < 1$ ,  $\mathcal{B}_m$  has a powerlike tail. In both cases,  $\mathcal{B}_m$  is a local weak solution in  $\mathbb{R}^n \times (0, \infty)$ . See Section 5 for the trace of  $\mathcal{B}_m$  at the initial time  $t = 0$ .

The definition of weak solutions and supersolutions does not include a time derivative of  $u$ . However, we would like to use test functions depending on  $u$ , and thus the time derivative  $\frac{\partial u}{\partial t}$  inevitably appears. To deal with this defect, a mollification procedure in the time direction, for instance Steklov averages or convolution with the standard mollifier, is usually employed. The mollification

$$(2.7) \quad u^*(x, t) = \frac{1}{\sigma} \int_0^t e^{(s-t)/\sigma} u(x, s) \, ds$$

is convenient. The aim is to obtain estimates independent of the time derivative of  $u^*$ , and then pass to the limit  $\sigma \rightarrow 0$ .

The basic properties of the mollification (2.7) are given in the following lemma, see [20].

**Lemma 2.3.** (1) *If  $u \in L^p(\Omega_T)$ , then*

$$\|u^*\|_{L^p(\Omega_T)} \leq \|u\|_{L^p(\Omega_T)},$$

$$(2.8) \quad \frac{\partial u^*}{\partial t} = \frac{u - u^*}{\sigma},$$

*and  $u^* \rightarrow u$  in  $L^p(\Omega_T)$  as  $\sigma \rightarrow 0$ .*

(2) *If  $\nabla u \in L^p(\Omega_T)$ , then  $\nabla(u^*) = (\nabla u)^*$ ,*

$$\|\nabla u^*\|_{L^p(\Omega_T)} \leq \|\nabla u\|_{L^p(\Omega_T)},$$

*and  $\nabla u^* \rightarrow \nabla u$  in  $L^p(\Omega_T)$  as  $\sigma \rightarrow 0$ .*

(3) *If  $u_k \rightarrow u$  in  $L^p(\Omega_T)$ , then also*

$$u_k^* \rightarrow u^* \text{ and } \frac{\partial u_k^*}{\partial t} \rightarrow \frac{\partial u^*}{\partial t}$$

*in  $L^p(\Omega_T)$ .*

(4) *If  $\nabla u_k \rightarrow \nabla u$  in  $L^p(\Omega_T)$ , then  $\nabla u_k^* \rightarrow \nabla u^*$  in  $L^p(\Omega_T)$ .*

(5) *Similar results hold for weak convergence in  $L^p(\Omega_T)$ .*

(6) *If  $\varphi \in C(\overline{\Omega_T})$ , then*

$$\varphi^*(x, t) + e^{-t/\sigma} \varphi(x, 0) \rightarrow \varphi(x, t)$$

*uniformly in  $\Omega_T$  as  $\sigma \rightarrow 0$ .*

We use the following estimate for the local version of our stability result. See [14, Lemma 2.15] or [19, Lemma 2.9] for the proof.

**Lemma 2.4.** *Let  $u$  be a weak solution such that  $0 \leq u \leq M < \infty$ , where  $M \geq 1$ , and let  $\eta$  be any nonnegative function in  $C_0^\infty(\Omega)$ . Then*

$$\int_{\Omega_T} \eta^2 |\nabla u^m|^2 dx dt \leq 2M^{m+1} \int_{\Omega} \eta^2 dx + 16M^{2m} \int_{\Omega_T} |\nabla \eta|^2 dx dt.$$

We use the following elementary lemma to pass pointwise convergences between various powers.

**Lemma 2.5.** *Let  $(f_i)$  be a sequence of positive functions on a measurable set  $E$  with finite measure such that*

$$f_i \rightarrow f \text{ in } L^1(E) \text{ and pointwise almost everywhere.}$$

*Assume that  $\alpha_i \rightarrow \alpha$  as  $i \rightarrow \infty$ . Then*

$$f_i^{\alpha_i} \rightarrow f^\alpha$$

*pointwise almost everywhere.*

*Proof.* By Egorov's Theorem, for any  $\varepsilon > 0$  there is a set  $F_\varepsilon$  such that  $|F_\varepsilon| < \varepsilon$  and  $f_i \rightarrow f$  uniformly in  $E \setminus F_\varepsilon$ . Pick any point  $x \in E \setminus F_\varepsilon$  such that

$$0 < \delta \leq f(x) \leq M < \infty.$$

Then

$$0 < \delta/2 \leq f_i(x) \leq 2M$$

for all sufficiently large  $i$  by uniform convergence. An application of the mean value theorem to the function  $\alpha \mapsto t^\alpha$  gives

$$\begin{aligned} |f_i(x)^{\alpha_i} - f(x)^\alpha| &\leq |f_i(x)^{\alpha_i} - f_i(x)^\alpha| + |f_i(x)^\alpha - f(x)^\alpha| \\ &\leq c(\delta, M)|\alpha_i - \alpha| + |f_i(x)^\alpha - f(x)^\alpha|. \end{aligned}$$

By the convergence assumptions, it follows that  $f_i(x)^{\alpha_i} \rightarrow f(x)^\alpha$  as  $i \rightarrow \infty$ .

Since  $\delta$ ,  $M$ , and  $\varepsilon$  are arbitrary, the above implies that  $f_i(x)^{\alpha_i} \rightarrow f(x)^\alpha$  for almost all  $x$  in the set  $\{0 < f(x) < \infty\}$ . For points where  $f(x) = 0$ , it is easy to check that  $f_i(x)^{\alpha_i} \rightarrow 0$ . Hence we have the desired convergence almost everywhere in the set  $\{f(x) < \infty\}$ , which is sufficient since  $f$  is integrable.  $\square$

### 3. STABILITY OF LOCAL WEAK SOLUTIONS

In this section, we establish a local version of stability for a bounded family of local weak solutions. This is beneficial since we may then apply the same result to both initial-boundary value problems in bounded domains and Cauchy problems on the whole space. The crucial point in our stability results is extracting pointwise convergent subsequences out of a sequence of solutions, in other words, a compactness property of solutions.

Recall that we use the notation  $U \Subset \Omega$  to mean that the closure  $\overline{U}$  is compact and contained in  $\Omega$ . In view of applying this result to the Cauchy problem on  $\mathbb{R}^n$ , we allow the cases  $\Omega = \mathbb{R}^n$  and  $T = \infty$ . We denote

$$m^+ = \sup_i m_i \quad \text{and} \quad m^- = \inf_i m_i,$$

with similar notations for other exponents.

The main result of this section is the following theorem.

**Theorem 3.1.** *Let  $m_i$ ,  $i = 1, 2, 3, \dots$ , be exponents such that*

$$(3.1) \quad m_i \rightarrow m \quad \text{as} \quad i \rightarrow \infty \quad \text{for some} \quad m > m_c = (n-2)_+/n$$

*Let  $u_i$ ,  $i = 1, 2, 3, \dots$ , be positive local weak solutions to*

$$\partial_t u_i - \Delta u_i^{m_i} = 0$$

in  $\Omega_T$ . Assume that we have the bound

$$(3.2) \quad \|u_i\|_{L^\infty(U_{t_1, t_2})} \leq M < \infty$$

in all cylinders  $U_{t_1, t_2}$  such that  $U \subseteq \Omega$  and  $0 < t_1 < t_2 < T$ .

Then there is a function  $u$  such that  $u_i \rightarrow u$  and  $u^{m_i} \rightarrow u^m$  pointwise almost everywhere in  $\Omega_T$ , for a subsequence still indexed by  $i$ . Further,  $u$  is a local weak solution to

$$\partial_t u - \Delta u^m = 0.$$

We split the proof of Theorem 3.1 into several lemmas. In view of the convergence assumption (3.1), we are free to assume that

$$m_c < m^- \quad \text{and} \quad m^+ < \infty.$$

Let us define the auxiliary exponents

$$m_i^\sharp = \max\{m_i, 1\} \quad \text{and} \quad m_i^\flat = \min\{m_i, 1\}.$$

Then  $m_i^\sharp \geq m_i^\flat$ , and one of these exponents always equals  $m_i$  and the other equals one. We use a similar notation for the limit exponent  $m$ .

The proof of Theorem 3.1 consists of three main steps: first we apply a compactness result [25] to certain auxiliary functions to find the limit function  $u$ . Then we show that  $u_i^{m_i^\sharp}$  converges to  $u^{m^\sharp}$  in measure. This is the most involved part of the proof, and for a key estimate we apply a test function due to Oleĭnik. In the last step, we establish the pointwise convergences  $u_i \rightarrow u$  and  $u_i^{m_i} \rightarrow u^m$  by applying Lemma 2.5, and the fact that  $u$  is a local weak solution by applying Lemma 2.4.

The next lemma will be used in proving the convergence in measure.

**Lemma 3.2.** *Let  $0 \leq s < t \leq M$ , where  $M \geq 1$ , be such that*

$$t^{m^\sharp} - s^{m^\sharp} \geq \lambda > 0.$$

*Then*

$$t^{m^\flat} - s^{m^\flat} \geq \frac{m^\flat}{m^\sharp} M^{m^\flat - m^\sharp} \min\{\lambda, \lambda^{m^\flat/m^\sharp}\}$$

*Proof.* Consider first the case  $s = 0$ . Then

$$t^{m^\flat} - s^{m^\flat} = t^{m^\flat} \geq \lambda^{m^\flat/m^\sharp},$$

since  $t^{m^\sharp} \geq \lambda$ . Assume then that  $s > 0$ . By the mean value theorem, we have

$$t^{m^\flat} - s^{m^\flat} = (t^{m^\sharp})^{m^\flat/m^\sharp} - (s^{m^\sharp})^{m^\flat/m^\sharp} = \frac{m^\flat}{m^\sharp} \xi^{m^\flat/m^\sharp - 1} (t^{m^\sharp} - s^{m^\sharp})$$



for some  $\xi \in (s^{m^\sharp}, t^{m^\sharp})$ . Since  $\frac{m^b}{m^\sharp} - 1 \leq 0$  and  $t^{m^\sharp} \leq M^{m^\sharp}$ , we have

$$\frac{m^b}{m^\sharp} \xi^{\frac{m^b}{m^\sharp}-1} (t^{m^\sharp} - s^{m^\sharp}) \geq \frac{m^b}{m^\sharp} M^{m^b-m^\sharp} \lambda,$$

which completes the proof.  $\square$

The following lemma provides the key estimate for showing that the original sequence also converges to the limit found by applying the compactness result.

**Lemma 3.3.** *Let  $U$  and  $u_i$  be as in Theorem 3.1, and let  $0 < t_1 < t_2 < T$ .*

*Fix a number  $0 < \delta < 1$ , and suppose that  $u_{i,\delta}$  is the unique function which satisfies*

$$\begin{cases} \partial_t u_{i,\delta} - \Delta u_i^{m_i} = 0 & \text{in } U_{t_1, t_2}, \\ u_{i,\delta}^{m_i} - u_i^{m_i} - \delta^{m_i} \in L^2(t_1, t_2; H_0^1(U)), \\ u_{i,\delta}(x, t_1) = u_i(x, t_1) + \delta, & x \in U. \end{cases}$$

*Then*

$$\int_{U_{t_1, t_2}} (u_{i,\delta} - u_i)(u_{i,\delta}^{m_i} - u_i^{m_i}) \, dx \, dt \leq c(\delta + \delta^{m^-}),$$

*where*

$$c = 2^{m^++1}(M^{m^+} + M + 1)|U_{t_1, t_2}|$$

*and  $M$  is the number appearing in (3.2).*

Before proceeding with the proof let us note that the technical reason for introducing the exponents  $m_i^\sharp$  and  $m_i^b$  is that we may write

$$(u_{i,\delta} - u_i)(u_{i,\delta}^{m_i} - u_i^{m_i}) = (u_{i,\delta}^{m_i^b} - u_i^{m_i^b})(u_{i,\delta}^{m_i^\sharp} - u_i^{m_i^\sharp})$$

when applying this lemma.

*Proof.* The proof is an application of a test function due to Oleřnik. The function  $u_{i,\delta}^{m_i} - u_i^{m_i} - \delta^{m_i}$  has zero boundary values in Sobolev's sense, so the same is true for the function

$$\eta(x, t) = \begin{cases} \int_t^{t_2} u_{i,\delta}^{m_i} - u_i^{m_i} - \delta^{m_i} \, ds, & t_1 < t < t_2, \\ 0, & t \geq t_2. \end{cases}$$

We use  $\eta$  as a test function in the equations satisfied by  $u_{i,\delta}$  and  $u_i$ , and subtract the results. This gives

$$\begin{aligned} & \int_{U_{t_1,t_2}} (u_{i,\delta} - u_i)(u_{i,\delta}^{m_i} - u_i^{m_i} - \delta^{m_i}) \, dx \, dt \\ & \quad + \int_{U_{t_1,t_2}} \nabla(u_{i,\delta}^{m_i} - u_i^{m_i}) \int_t^{t_2} \nabla(u_{i,\delta}^{m_i} - u_i^{m_i}) \, ds \, dx \, dt \\ & = \delta \int_U \int_{t_1}^{t_2} u_{i,\delta}^{m_i} - u_i^{m_i} \, ds \, dx. \end{aligned}$$

We move the term with  $\delta$  to the right hand side, and integrate with respect to  $t$  in the elliptic term. We get

$$\begin{aligned} & \int_{U_{t_1,t_2}} (u_{i,\delta} - u_i)(u_{i,\delta}^{m_i} - u_i^{m_i}) \, dx \, dt + \frac{1}{2} \int_U \left[ \int_{t_1}^{t_2} \nabla(u_{i,\delta}^{m_i} - u_i^{m_i}) \, ds \right]^2 \, dx \\ & = \delta^{m_i} \int_{U_{t_1,t_2}} (u_{i,\delta} - u_i) \, dx \, dt + \delta \int_{U_{t_1,t_2}} (u_{i,\delta}^{m_i} - u_i^{m_i}) \, dx \, dt \end{aligned}$$

Now, both of the terms on the left are positive, so we have the freedom to take absolute values of the right hand side. The claim then follows by discarding the second term on the left hand side, and estimating the integrals on the last line by using (3.2) and the fact that  $u_{i,\delta} \leq M+1$ , by the comparison principle. Indeed, we have

$$\left| \int_{U_{t_1,t_2}} (u_{i,\delta} - u_i) \, dx \, dt \right| \leq 2(M+1)|U_{t_1,t_2}|$$

and

$$\left| \int_{U_{t_1,t_2}} (u_{i,\delta}^{m_i} - u_i^{m_i}) \, dx \, dt \right| \leq 2^{m^+} (M^{m^+} + 1) |U_{t_1,t_2}|. \quad \square$$

The following lemma is the key step in the proof of Theorem 3.1.

**Lemma 3.4.** *There exists a subsequence, still indexed by  $i$ , and a function  $u$  such that*

$$u_i^{m_i^\#} \rightarrow u^{m^\#} \quad \text{as } i \rightarrow \infty$$

*in measure and pointwise almost everywhere in  $U_{t_1,t_2}$ .*

*Proof.* Recall that  $U$  is an open set such that  $U \Subset \Omega$ , and  $0 < t_1 < t_2 < T$ . We use the auxiliary functions  $u_{i,\delta}$  defined in Lemma 3.3: for any  $0 < \delta < 1$ ,  $u_{i,\delta}$  is the weak solution in  $U_{t_1,t_2}$  with boundary and initial values  $u_i^{m_i} + \delta^{m_i}$  and  $u(\cdot, t_1) + \delta$ , respectively. Note that  $\delta \leq u_{i,\delta} \leq M+1$  by the comparison principle.

Fix then a regular open set  $V \Subset U$ , and  $t_1 < s_1 < s_2 < t_2$ . For indices such that  $m_i \geq 1$ , we have

$$\begin{aligned} |\nabla u_{i,\delta}| &= \nabla(u_{i,\delta}^{m_i})^{1/m_i} = |u_i^{1/m_i-1} \nabla u_{i,\delta}^{m_i}| \\ &\leq \delta^{1/m_i-1} |\nabla u_{i,\delta}^{m_i}| \leq \delta^{1/m^+-1} |\nabla u_{i,\delta}^{m_i}|, \end{aligned}$$

and for indices such that  $m_i < 1$  we have

$$\begin{aligned} |\nabla u_{i,\delta}| &= |\nabla(u_{i,\delta}^{m_i})^{1/m_i}| \leq |u_i^{1/m_i-1} \nabla u_{i,\delta}^{m_i}| \\ &\leq (M + \delta)^{1/m_i-1} |\nabla u_{i,\delta}^{m_i}| \leq (M + 1)^{1/m^--1} |\nabla u_{i,\delta}^{m_i}|. \end{aligned}$$

Thus Lemma 2.4 implies that  $(\nabla u_{i,\delta})$  is bounded in  $L^2(V_{s_1,s_2})$ . The fact that the sequence  $(\partial_t u_{i,\delta})$  is bounded in  $L^2(s_1, s_2; H^{-1}(V))$  follows easily from the equation satisfied by  $u_i, \delta$  and Lemma 2.4. An application of [25, Corollary 4] shows that the sequence  $(u_{i,\delta})$  is compact in  $L^2(V_{s_1,s_2})$ . Thus for any fixed  $\delta > 0$ , we may extract a subsequence which converges in  $L^2(V_{s_1,s_2})$  and pointwise almost everywhere.

The next step is a repeated application of the compactness established in the previous step. Let  $\delta_j = 1/j$ ,  $j = 1, 2, 3, \dots$ , and for  $j = 1$  pick indices  $i$  such that

$$u_{i,\delta_1} \rightarrow u_{\delta_1}$$

for some function  $u_{\delta_1}$ , the convergence being in  $L^2(V_{s_1,s_2})$  and pointwise almost everywhere. We proceed by picking a further subsequence so that also

$$u_{i,\delta_2} \rightarrow u_{\delta_2}$$

in  $L^2(V_{s_1,s_2})$  and pointwise almost everywhere. We continue this way, and get a twodimensional table of indices  $(i, j)$ , where the values of  $i$  in the diagonal positions have the property that

$$u_{i,\delta_j} \rightarrow u_{\delta_j} \text{ as } i \rightarrow \infty \text{ for all } j, \text{ in } L^2(V_{s_1,s_2}) \text{ and pointwise a.e.}$$

By the comparison principle, we have

$$u_{i,\delta_j} \geq u_{i,\delta_k} \quad \text{for } k \geq j,$$

and letting  $i$  tend to infinity we get

$$u_{\delta_j} \geq u_{\delta_k}, \quad \text{whenever } k \geq j.$$

Thus we may define the function  $u$  in the claim of the current lemma as the pointwise limit

$$u(x, t) = \lim_{j \rightarrow \infty} u_{\delta_j}(x, t).$$

We have now found the limit function  $u$ , and the proof will be completed by showing that  $u_i^{m_i^\sharp}$  converges to  $u^{m^\sharp}$  in measure, first in  $V_{s_1,s_2}$  and then in  $U_{t_1,t_2}$  by an exhaustion argument. The first convergence is

a consequence of Lemmas 3.2 and 3.3. Obviously  $u_{\delta_j}^{m^\sharp} \rightarrow u^{m^\sharp}$  pointwise almost everywhere as  $j \rightarrow \infty$ . Further, Lemma 2.5 implies that also  $u_{i,\delta_j}^{m_i^\sharp} \rightarrow u_{\delta_j}^{m^\sharp}$  a.e. as  $i \rightarrow \infty$ .

To show the convergence in measure, let  $\lambda > 0$  and  $\varepsilon > 0$ . We have

$$\begin{aligned} |\{|u_i^{m_i^\sharp} - u^{m^\sharp}| \geq \lambda\}| &\leq |\{|u_i^{m_i^\sharp} - u_{i,\delta_j}^{m_i^\sharp}| \geq \lambda/3\}| \\ &\quad + |\{|u_{i,\delta_j}^{m_i^\sharp} - u_{\delta_j}^{m^\sharp}| \geq \lambda/3\}| + |\{|u_{\delta_j}^{m^\sharp} - u^{m^\sharp}| \geq \lambda/3\}| \end{aligned}$$

Since  $u_{i,\delta_j}^{m_i^\sharp} - u_i^{m_i^\sharp} \geq c \min\{\lambda, \lambda^{m_i^\sharp/m_i^\sharp}\}$  by Lemma 3.2, we have

$$\begin{aligned} |\{|u_i^{m_i^\sharp} - u_{i,\delta_j}^{m_i^\sharp}| \geq \lambda/3\}| &= \int_{\{|u_i^{m_i^\sharp} - u_{i,\delta_j}^{m_i^\sharp}| \geq \lambda/3\}} 1 \, dx \, dt \\ &\leq \frac{3}{\lambda} \int_{\{|u_i^{m_i^\sharp} - u_{i,\delta_j}^{m_i^\sharp}| \geq \lambda/3\}} (u_{i,\delta_j}^{m_i^\sharp} - u_i^{m_i^\sharp}) \, dx \, dt \\ &\leq \frac{c}{\lambda \min\{\lambda, \lambda^{m_i^\sharp/m_i^\sharp}\}} \int_{V_{s_1, s_2}} (u_{i,\delta_j}^{m_i^\sharp} - u_i^{m_i^\sharp})(u_{i,\delta_j}^{m_i^\sharp} - u_i^{m_i^\sharp}) \, dx \, dt \\ &\leq \frac{c}{\lambda \min\{\lambda, \lambda^{m_i^\sharp/m_i^\sharp}\}} (\delta_j + \delta_j^{m^-}), \end{aligned}$$

where the last inequality follows from Lemma 3.3 and the fact that

$$(u_{i,\delta} - u_i)(u_{i,\delta}^{m_i^\sharp} - u_i^{m_i^\sharp}) = (u_{i,\delta}^{m_i^\sharp} - u_i^{m_i^\sharp})(u_{i,\delta}^{m_i^\sharp} - u_i^{m_i^\sharp}).$$

Thus for each fixed  $\lambda > 0$ , we may choose  $\delta_j$  small enough, so that

$$|\{|u_{i,\delta_j}^{m_i^\sharp} - u_i^{m_i^\sharp}| \geq \lambda/3\}| \leq \varepsilon.$$

Here it is crucial that this choice can be made independent of  $i$ .

From the pointwise convergence established above, it follows that  $u_{\delta_j}^{m^\sharp} \rightarrow u^{m^\sharp}$  in measure. Thus we may choose  $\delta_j$  small enough, so that

$$|\{|u_{\delta_j}^{m^\sharp} - u^{m^\sharp}| \geq \lambda/3\}| \leq \varepsilon.$$

Finally, we know that  $u_{i,\delta_j}^{m_i^\sharp} \rightarrow u_{\delta_j}^{m^\sharp}$  pointwise almost everywhere and hence also in measure. Thus the above estimates imply that

$$\limsup_{i \rightarrow \infty} |\{|u_i^{m_i^\sharp} - u^{m^\sharp}| \geq \lambda\}| \leq \lim_{i \rightarrow \infty} |\{|u_{i,\delta_j}^{m_i^\sharp} - u_{\delta_j}^{m^\sharp}| \geq \lambda/3\}| + 2\varepsilon = 2\varepsilon.$$

Since  $\varepsilon$  was arbitrary, we get

$$\lim_{i \rightarrow \infty} |\{|u_i^{m_i^\sharp} - u^{m^\sharp}| \geq \lambda\}| = 0$$

for all  $\lambda > 0$ , as desired. We have proved that for cylinders  $V_{s_1, s_2}$  such that  $V_{s_1, s_2} \subseteq U_{t_1, t_2} \subseteq \Omega_T$ , we may extract a subsequence such that

$$u_i^{m^\sharp} \rightarrow u^{m^\sharp} \quad \text{in measure in } V_{s_1, s_2} \text{ as } i \rightarrow \infty.$$

Passing to a further subsequence, we have convergence pointwise almost everywhere.

To find a subsequence converging in the whole of  $U_{t_1, t_2}$ , we use a diagonalization argument. Exhaust  $U$  by regular open sets  $V^k$ , and choose nested time intervals  $(s_1^1, s_2^1) \subseteq (s_1^2, s_2^2) \subseteq \dots$ ,  $k = 1, 2, \dots$ , such that

$$U_{t_1, t_2} = \bigcup_{k=1}^{\infty} V_{s_1^k, s_2^k}^k.$$

First, pick a subsequence such that  $(u_i^{m^\sharp})$  converges in measure in  $V_{s_1^1, s_2^1}^1$  to  $u^{m^\sharp}$ . The procedure continues inductively, by the selection of a further subsequence that converges in measure in  $V_{s_1^{k+1}, s_2^{k+1}}^{k+1}$  to the function  $u^{m^\sharp}$ . Taking the  $k$ th index in the subsequence selected in the  $k$ th step yields a subsequence convergent in measure in  $U_{t_1, t_2}$ .  $\square$

The next step in the proof of Theorem 3.1 is to prove the convergence of the other powers of  $u$  by using the convergence established in the previous lemma.

**Lemma 3.5.** *Let  $u_i$ ,  $i = 1, 2, 3, \dots$ , be such that  $0 \leq u_i \leq M < \infty$ , and  $u_i^{m^\sharp} \rightarrow u^{m^\sharp}$  in measure in  $U_{t_1, t_2}$ . Then  $u_i \rightarrow u$  and  $u_i^{m_i} \rightarrow u^m$  almost everywhere in  $U_{t_1, t_2}$  as  $i \rightarrow \infty$ .*

*Proof.* The bound  $u_i \leq M$  and convergence in measure imply that  $u_i^{m_i^\sharp} \rightarrow u^{m^\sharp}$  in  $L^q(U_{t_1, t_2})$  for any finite  $q$ . Thus we obtain the desired convergences by two applications of Lemma 2.5, first passing from the convergence  $u_i^{m_i^\sharp} \rightarrow u^{m^\sharp}$  to the convergence  $u_i \rightarrow u$ , and then from  $u_i \rightarrow u$  to  $u_i^{m_i} \rightarrow u^m$ .  $\square$

With all of the preceding lemmas available, the proof of Theorem 3.1 is now a relatively simple matter.

*Proof of Theorem 3.1.* The pointwise convergences follow from Lemma 3.5 by an exhaustion argument, as in the proof of Lemma 3.4. To show that  $u$  is a local weak solution, fix a test function  $\varphi \in C_0^\infty(\Omega_T)$ . We use the bound  $u_i \leq M$  and Lemma 2.4 to get subsequences of  $(u_i)$  and  $(\nabla u_i^{m_i})$ , weakly convergent in  $L^2(\text{supp } \varphi)$ . Due to pointwise convergences, we see that the weak limits must be  $u$  and  $\nabla u^m$ , respectively.

From the weak convergences, it follows that

$$\int_{\Omega_T} -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \, dx \, dt = \lim_{i \rightarrow \infty} \left( \int_{\Omega_T} -u_i \frac{\partial \varphi}{\partial t} - \nabla u_i^{m_i} \cdot \nabla \varphi \, dx \, dt \right) = 0.$$

This holds for all test functions  $\varphi$ , so the proof is complete.  $\square$

#### 4. STABILITY OF DIRICHLET PROBLEMS

In this section we prove a stability result for Dirichlet boundary value problems in bounded domains. We use Theorem 3.1, a local  $L^\infty$  estimate (Proposition 4.4), and a simple energy estimate (Lemma 4.3). In this section, we take  $\Omega$  to be bounded.

**Definition 4.1.** Let  $u_0 \in L^{m+1}(\Omega)$  and  $g \in H^1(0, T; H^1(\Omega))$ . A positive function  $u$  such that  $u^m \in L^2(0, T; H^1(\Omega))$  is a solution of the initial-boundary value problem

$$(4.1) \quad \begin{cases} \partial_t u - \Delta u^m = 0, & \text{in } \Omega_T, \\ u^m = g, & \text{on } \partial\Omega \times [0, T], \\ u(x, 0) = u_0(x). \end{cases}$$

if  $u^m - g \in L^2(0, T; H_0^1(\Omega))$ , and

$$\int_{\Omega_T} -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \, dx \, dt + \int_{\Omega} u(x, T) \varphi(x, T) \, dx = \int_{\Omega} u_0(x) \varphi(x, 0) \, dx$$

for all smooth test functions  $\varphi$  which vanish on the lateral boundary of  $\Omega_T$ .

For the existence and uniqueness of solutions in the above sense, see [27, Chapter 5]. By the usual approximation argument, we may use test functions  $\varphi \in L^2(0, T; H_0^1(\Omega))$ .

Let  $u_i$  be the solution to

$$(4.2) \quad \begin{cases} \partial_t u_i - \Delta u_i^{m_i} = 0, & \text{in } \Omega_T, \\ u_i^{m_i} = g, & \text{on } \partial\Omega \times [0, T], \\ u_i(x, 0) = u_0(x) \end{cases}$$

in the sense of definition 4.1. We will find a subsequence of  $(u_i)$  that converges in a suitable sense to a function  $u$ , and show that  $u$  is a solution of the limit problem, i.e. satisfies

$$(4.3) \quad \begin{cases} \partial_t u - \Delta u^m = 0, & \text{in } \Omega_T, \\ u^m = g, & \text{on } \partial\Omega \times [0, T], \\ u(x, 0) = u_0(x). \end{cases}$$

More precisely, we have the following theorem.

**Theorem 4.2.** *Let  $m_i$ ,  $i = 1, 2, 3, \dots$ , be a sequence of exponents such that*

$$m_i \rightarrow m > m_c = (n-2)_+/n \quad \text{as } i \rightarrow \infty$$

*Let  $u_i$  be the solutions to (4.2) with fixed initial and boundary values  $g$  and  $u_0$ , where*

$$g \in H^1(0, T; H^1(\Omega)), \quad \frac{\partial g}{\partial t} \in L^{1+1/m^-}(\Omega_T), \quad \text{and} \quad u_0 \in L^{m^++1}(\Omega).$$

*Finally, let  $u$  be the solution to (4.3) with the boundary and initial values  $g$  and  $u_0$ , respectively.*

*Then*

- (1)  $u_i \rightarrow u$  in  $L^q(\Omega_T)$  for all  $1 \leq q < 1 + m$ .
- (2)  $u_i^{m_i} \rightarrow u^m$  in  $L^s(\Omega_T)$  for all  $1 \leq s < 2\kappa$ , where

$$\kappa = 1 + \frac{1}{m} + \frac{1}{mn}.$$

- (3)  $\nabla u_i^{m_i} \rightarrow \nabla u^m$  weakly in  $L^2(\Omega_T)$ .

We need an energy estimate for establishing that the limit function attains the right boundary values in Sobolev's sense, and for verifying the local boundedness assumption in Theorem 3.1. To derive it, we use the equation satisfied by the mollified solution  $u^*$ :

$$(4.4) \quad \int_{\Omega_T} \varphi \frac{\partial u^*}{\partial t} + \nabla(u^m)^* \cdot \nabla \varphi \, dx \, dt = \int_{\Omega} u_0(x) \left( \frac{1}{\sigma} \int_0^T \varphi e^{-s/\sigma} \, ds \right) \, dx$$

This is required to hold for all test functions  $\varphi \in L^2(0, T; H_0^1(\Omega))$ . The equation (4.4) follows from (4.1) by straightforward manipulations involving a change of variables and Fubini's theorem.

**Lemma 4.3.** *Let  $u$  be the weak solution with boundary values  $g$  and initial values  $u_0$ . Then*

$$(4.5) \quad \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} u^{m+1}(x, t) \, dx + \int_{\Omega_T} |\nabla u^m|^2 \, dx \, dt \\ \leq c \left( \int_{\Omega_T} |\nabla g|^2 + \left| \frac{\partial g}{\partial t} \right|^{1+1/m} \, dx \, dt + \int_{\Omega} u_0(x)^{m+1} \, dx \right).$$

*Proof.* We test the regularized equation (4.4) with  $\varphi = u^m - g$ . This yields

$$\int_{\Omega_T} \frac{\partial u^*}{\partial t} (u^m - g) \, dx \, dt + \int_{\Omega_T} \nabla(u^m)^* \cdot \nabla(u^m - g) \, dx \, dt \\ = \int_{\Omega} u_0(x) \left( \frac{1}{\sigma} \int_0^T (u^m - g) e^{-s/\sigma} \, ds \right) \, dx.$$

We rearrange this to get

$$\begin{aligned}
 (4.6) \quad & \int_{\Omega_T} \frac{\partial u^*}{\partial t} u^m \, dx \, dt + \int_{\Omega_T} \nabla(u^m)^* \cdot \nabla u^m \, dx \, dt \\
 &= \int_{\Omega_T} \frac{\partial u^*}{\partial t} g \, dx \, dt + \int_{\Omega_T} \nabla(u^m)^* \cdot \nabla g \, dx \, dt \\
 &\quad + \int_{\Omega} u_0(x) \left( \frac{1}{\sigma} \int_0^T (u^m - g) e^{-s/\sigma} \, ds \right) \, dx.
 \end{aligned}$$

For the first term on the left, we get

$$\begin{aligned}
 \int_{\Omega_T} \frac{\partial u^*}{\partial t} u^m \, dx \, dt &= \int_{\Omega_T} \frac{\partial u^*}{\partial t} (u^*)^m \, dx \, dt + \int_{\Omega_T} \frac{\partial u^*}{\partial t} (u^m - (u^*)^m) \, dx \, dt \\
 &\geq \int_{\Omega_T} \frac{\partial u^*}{\partial t} (u^*)^m \, dx \, dt = \int_{\Omega} \frac{u^*(x, T)^{m+1}}{m+1} \, dx \\
 &\rightarrow \int_{\Omega} \frac{u(x, T)^{m+1}}{m+1} \, dx
 \end{aligned}$$

as  $\sigma \rightarrow 0$ . Here we used (2.8) to get the inequality. For the second, it suffices to note that

$$\int_{\Omega_T} \nabla(u^m)^* \cdot \nabla u^m \, dx \, dt \rightarrow \int_{\Omega_T} |\nabla u^m|^2 \, dx \, dt$$

as  $\sigma \rightarrow 0$ .

The limits of the left hand side terms in (4.6) are positive, so we are free to take absolute values on the right after passing to the limit  $\sigma \rightarrow 0$ . In the first term on the left, we integrate by parts before taking the limit:

$$\int_{\Omega_T} \frac{\partial u^*}{\partial t} g \, dx \, dt = - \int_{\Omega_T} u^* \frac{\partial g}{\partial t} \, dx \, dt \rightarrow - \int_{\Omega_T} u \frac{\partial g}{\partial t} \, dx \, dt.$$

We proceed by taking absolute values, applying Young's inequality, and taking the supremum over  $t$ . We get

$$\left| \int_{\Omega_T} u \frac{\partial g}{\partial t} \, dx \, dt \right| \leq \varepsilon \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} u^{1+m}(x, t) \, dx + c_{\varepsilon} \int_{\Omega_T} \left| \frac{\partial g}{\partial t} \right|^{1+1/m} \, dx \, dt.$$

The limit of the second term on the right of (4.6) is

$$\int_{\Omega_T} \nabla u^m \cdot \nabla g \, dx \, dt.$$

Here we simply take absolute values and use Young's inequality to get

$$\left| \int_{\Omega_T} \nabla u^m \cdot \nabla g \, dx \, dt \right| \leq \varepsilon \int_{\Omega_T} |\nabla u^m|^2 \, dx \, dt + c \int_{\Omega_T} |\nabla g|^2 \, dx \, dt.$$



For the third term on the right of (4.6), taking the limit yields

$$\int_{\Omega} u_0^{m+1}(x) \, dx - \int_{\Omega} u_0(x) g(x, 0) \, dx;$$

The first term needs no further estimations, and the second is negative so we may discard it.

We have arrived at

$$(4.7) \quad \int_{\Omega} \frac{u^{m+1}(x, T)}{m+1} \, dx + \int_{\Omega_T} |\nabla u^m|^2 \, dx \, dt \\ \leq \varepsilon \int_{\Omega_T} |\nabla u^m|^2 \, dx \, dt + \varepsilon \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} u(x, t)^{1+m} \, dx \\ + c \int_{\Omega} u_0^{m+1}(x) \, dx + c \left( \int_{\Omega_T} |\nabla g|^2 + \left| \frac{\partial g}{\partial t} \right|^{1+1/m} \, dx \, dt \right).$$

To finish the proof, replace  $T$  in the above proof by a number  $0 < \tau < T$  such that

$$\int_{\Omega} u(x, \tau)^{m+1} \, dx \geq \frac{1}{2} \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} u(x, t)^{m+1} \, dx.$$

This leads to an estimate for

$$\operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} u(x, t)^{m+1} \, dx + \int_{\Omega_T} |\nabla u^m|^2 \, dx \, dt$$

in terms of the right hand side of (4.7). The claim follows by choosing  $\varepsilon$  sufficiently small and then absorbing the matching terms to the left hand side.  $\square$

An analysis of the above proof shows that the constant may be taken to depend only on  $m^+$  and  $m^-$ , as  $m$  varies over the interval  $[m^-, m^+]$ . Thus, in view of (4.5) and the assumptions on  $g$  and  $u_0$  in Theorem 4.2, we see that the sequence  $(\nabla u_i^{m_i})$  is bounded in  $L^2(\Omega_T)$ , and by the Sobolev embedding,  $(u_i^{m_i})$  is bounded in  $L^2(\Omega_T)$ .

We use the following estimate to verify the local boundedness assumption in Theorem 3.1. See [9, Proposition B.5.1].

**Proposition 4.4.** *Let  $u$  be a local weak solution, and suppose that  $B_{\rho} \times [t_0 - \rho^2, t_0] \Subset \Omega_T$ . Then*

$$\operatorname{ess\,sup}_{B_{\rho/2} \times [t_0 - \rho^2/2, t_0]} u \leq c \left( \int_{B_{\rho} \times [t_0 - \rho^2, t_0]} u \, dx \, dt \right)^{2/\lambda} + 1,$$

where

$$\lambda = n(m-1) + 2 > 0.$$

As is discussed in [9], the constants in estimates of this type are stable as  $m$  decreases or increases to one, but blow up as  $m \rightarrow m_c$  or  $m \rightarrow \infty$ . Hence we may again assume that the constant is independent of  $i$ , as  $m_i$  varies in the interval  $[m^-, m^+]$ . Note also that we have written this estimate over standard parabolic cylinders  $B_\rho \times (t_0 - \rho^2, t_0)$ , for otherwise the estimate would not be stable.

The proof of our stability result is now a straightforward matter.

*Proof of Theorem 4.2.* We combine Lemmas 2.1 and 4.3 with Proposition 4.4 to verify the local boundedness assumption (3.2) in Theorem 3.1. Thus the theorem yields a subsequence and a function  $\tilde{u}$  such that  $u_i \rightarrow \tilde{u}$  and  $u_i^{m_i} \rightarrow \tilde{u}^m$ , pointwise almost everywhere in  $\Omega_T$ . We may also assume that  $\nabla u_i^{m_i} \rightarrow \nabla \tilde{u}^m$  weakly in  $L^2(\Omega_T)$ , by Lemma 4.3 and the pointwise convergence.

To pass to the limit in the equations, we need a bound on  $u_i$ . We have

$$\begin{aligned} \int_{\Omega_T} u_i^{1+m^-} dx dt &\leq c \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} u^{1+m^-}(x, t) dx \\ &\leq c \left( \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} u_i^{1+m_i}(x, t) dx + 1 \right). \end{aligned}$$

This together with Lemma 4.3 implies that the sequence  $(u_i)$  is bounded in  $L^{1+m^-}(\Omega_T)$ . By reflexivity, we may assume that  $u_i \rightarrow \tilde{u}$  weakly.

By applying the weak convergences, we see that  $\tilde{u}$  must be a solution of the limit equation with the right boundary and initial values. First, it is clear that

$$\tilde{u}^m - g \in L^2(0, T; H_0^1(\Omega)),$$

since  $L^2(0, T; H_0^1(\Omega))$  is weakly closed, by virtue of being a closed subspace of  $L^2(0, T; H^1(\Omega))$ . Further, we have

$$\begin{aligned} \int_{\Omega_T} -\tilde{u} \frac{\partial \varphi}{\partial t} + \nabla \tilde{u}^m \cdot \nabla \varphi dx dt &= \lim_{i \rightarrow \infty} \int_{\Omega_T} -u_i \frac{\partial \varphi}{\partial t} + \nabla u_i^{m_i} \cdot \nabla \varphi dx dt \\ &= \int_{\Omega} u_0(x) \varphi(x, 0) dx \end{aligned}$$

for all smooth test functions  $\varphi$  vanishing on the lateral boundary and at  $t = T$  by the weak convergences; by uniqueness of weak solutions, this means that  $\tilde{u} = u$ .

Convergence in measure and a bound in  $L^p$  imply convergence in  $L^r$  for any  $r < p$ . For the pointwise convergent subsequences, the claims

$$u_i \rightarrow u \quad \text{in} \quad L^q(\Omega_T), \quad 1 \leq q < 1 + m,$$

and

$$u_i^{m_i} \rightarrow u^m \quad \text{in } L^s(\Omega_T), \quad 1 \leq s < 2\kappa$$

follow from this. The corresponding convergence of the original sequence then follows from the fact that any subsequence that converges, must converge to the same limit, by the uniqueness of weak solutions. This also holds for weak convergence, whence we get the remaining claim about the weak convergence of the gradients.  $\square$

*Remark 4.5.* Generalizing Theorem 4.2 to, e.g., Neumann boundary conditions is a matter of proving a suitable counterpart of the energy estimate of Lemma 4.3. Indeed, Proposition 4.4 is a purely local estimate, independent of any boundary conditions. We leave the details to the interested reader.

## 5. STABILITY OF CAUCHY PROBLEMS

In this section, we prove stability of Cauchy problems on the whole space  $\mathbb{R}^n$ . As for Dirichlet problems, Theorem 3.1 is the key tool. Other results we use are the  $L^1$  contraction property, and the  $L^1 - L^\infty$  regularizing effect

**Definition 5.1.** Let  $\mu$  be a positive Borel measure on  $\mathbb{R}^n$  such that

$$\mu(\mathbb{R}^n) < \infty.$$

A positive function  $u : \mathbb{R}^n \times (0, \infty) \rightarrow [0, \infty)$  is a solution to the initial value problem

$$\begin{cases} u_t - \Delta u^m = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = \mu \end{cases}$$

if the following hold.

- (1)  $u \in L^\infty(0, \infty; L^1(\mathbb{R}^n))$ ,  $u \in L^\infty(\mathbb{R}^n \times (\tau, \infty))$  for all  $\tau > 0$ , and  $u^m \in L^1(S \times (0, T))$  for all compact subsets  $S$  of  $\mathbb{R}^n$  and finite  $T$ .
- (2)  $u$  is a local weak solution in  $\mathbb{R}^n \times (0, \infty)$  in the sense of Definition 2.2.
- (3) For all test functions  $\varphi \in C_0^\infty(\mathbb{R}^n \times [0, \infty))$ , it holds that

$$(5.1) \quad \int_{\mathbb{R}^n \times [0, \infty)} -u \frac{\partial \varphi}{\partial t} - u^m \Delta \varphi \, dx \, dt = \int_{\mathbb{R}^n} \varphi(x, 0) \, d\mu$$

Solutions with the above properties exist and are unique. Indeed, by approximating  $\mu$  with nice initial data, we can construct solutions such that the estimates of Theorem 5.3 below hold, and these estimates give the properties of  $u$  in Definition 5.1. See [6, 27] for the details. For the

uniqueness, we note that (5.1) and the integrability of  $u^m$  up to the initial time imply that

$$(5.2) \quad \int_{\mathbb{R}^n} u(x, \tau) \eta(x) dx \rightarrow \int_{\mathbb{R}^n} \eta(x) d\mu \quad \text{as } \tau \rightarrow 0.$$

for all  $\eta \in C_0^\infty(\mathbb{R}^n)$ ; thus we may appeal to the uniqueness results in [11, 23]. See also [6] for an account of this uniqueness theory.

The Barenblatt solutions (2.5) and (2.6) furnish examples of a solution to the Cauchy problem in the sense of Definition 5.1. One usually normalizes  $\mathcal{B}_m$  by choosing the constant  $C$  in (2.5), (2.6) so that

$$\int_{\Omega} \mathcal{B}_m(x, t) dx = 1$$

for all  $t > 0$ . With this normalization,  $\mathcal{B}_m$  is the unique solution to the Cauchy problem with initial trace  $\mu$  given by Dirac's delta at the origin, as a straightforward computation shows.

For Cauchy problems, we have the following stability result.

**Theorem 5.2.** *Let  $m_i$ ,  $i = 1, 2, 3, \dots$ , be a sequence of exponents such that*

$$m_i \rightarrow m > m_c = (n - 2)_+/n \quad \text{as } i \rightarrow \infty,$$

*and let  $u_i$  be the solutions to*

$$(5.3) \quad \begin{cases} \partial_t u_i - \Delta u_i^{m_i} = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u_i(x, 0) = \mu \end{cases}$$

*with fixed initial trace  $\mu$ . Further, let  $u$  be the solution to the limit problem*

$$(5.4) \quad \begin{cases} \partial_t u - \Delta u^m = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u_i(x, 0) = \mu \end{cases}$$

*with the same initial trace  $\mu$ .*

*Then for all compact sets  $S$  in  $\mathbb{R}^n$  and all finite  $T$ , we have*

- (1)  $u_i \rightarrow u$  in  $L^q(S_T)$  for all  $1 \leq q < m + 2/n$ ,
- (2)  $u_i^{m_i} \rightarrow u^m$  in  $L^s(S_T)$  for all  $1 \leq s < 1 + 2/mn$ ,
- (3)  $\nabla u_i^{m_i} \rightarrow \nabla u^m$  weakly in  $L_{\text{loc}}^2(\mathbb{R}^n \times (0, \infty))$ .

The following theorem provides the necessary estimates for our stability result. The admissible range of the integrability exponent  $q$  in (5.7) is sharp for the Barenblatt solution. This can be checked by a simple computation.

**Theorem 5.3.** *Let  $u \geq 0$  be a solution to the Cauchy problem with initial data  $\mu$  such that*

$$\|\mu\| = \mu(\mathbb{R}^n) < \infty.$$

*Then the following estimates hold.*

*For all  $t > 0$ , we have*

$$(5.5) \quad \|u(\cdot, t)\|_{L^1(\mathbb{R}^n)} \leq \|\mu\|.$$

*For every  $t > 0$ , we have*

$$(5.6) \quad u(x, t) \leq c\|\mu\|^{2/\lambda} t^{-n/\lambda}.$$

*where*

$$\lambda = n(m-1) + 2.$$

*The function  $u^m$  belongs to  $L^q(S_T)$  for all compact sets  $S$  in  $\mathbb{R}^n$  and all finite  $T$ , for*

$$1 \leq q < 1 + \frac{2}{mn}.$$

*We also have the estimate*

$$(5.7) \quad \int_{S_T} u^{mq} \, dx \, dt \leq c\|\mu\|^{\frac{2}{\lambda}(mq-1)+1} T^{-\frac{n}{\lambda}(mq-1)+1}.$$

*Proof.* The inequalities (5.5) and (5.6) are standard estimates for the Cauchy problem, see [6, 26, 27]. The inequality (5.7) is a slight refinement of the well-known fact that  $u^m$  is integrable up to the initial time locally in space, and follows from the first two. For the reader's convenience, we present the computation here. By applying (5.6) in the first inequality and (5.5) in the second, we have

$$\begin{aligned} \int_{S_T} u^{mq} \, dx \, dt &\leq c\|\mu\|^{\frac{2}{\lambda}(mq-1)} \int_{S_T} u(x, t) t^{-\frac{n}{\lambda}(mq-1)} \, dx \, dt \\ &\leq c\|\mu\|^{\frac{2}{\lambda}(mq-1)+1} \int_0^T t^{-\frac{n}{\lambda}(mq-1)} \, dt. \end{aligned}$$

We may evaluate the integral with respect to time and obtain (5.7) if

$$-\frac{n}{\lambda}(mq-1) > -1,$$

which is equivalent with

$$q < 1 + \frac{2}{mn}. \quad \square$$

The sharp constants in (5.6), as  $m$  varies, are given in [26, p. 26]. The situation is similar to that of Proposition 4.4: the constants are stable as  $m$  either increases or decreases to one, but blow up as  $m \rightarrow m_c$

or  $m \rightarrow \infty$ . Thus we are again free to assume that the constants are independent of  $i$  as  $m_i$  varies in the interval  $[m^-, m^+]$ .

*Proof of Theorem 5.2.* We use (5.6) to conclude that the sequence  $(u_i)$  is locally bounded in  $\mathbb{R}^n \times (0, \infty)$ . Thus Theorem 3.1 gives us point-wise convergent subsequences of  $(u_i)$  and  $(u_i^{m_i})$ , with limits  $\tilde{u}$  and  $\tilde{u}^m$ , respectively.

Convergence in measure and a bound in  $L^p$  imply convergence in  $L^r$  for any  $r < p$ . From this, the claims

$$u_i \rightarrow \tilde{u} \quad \text{in} \quad L^q(S_T), \quad 1 \leq q < m + 2/n,$$

and

$$u_i^{m_i} \rightarrow \tilde{u}^m \quad \text{in} \quad L^s(S_T), \quad 1 \leq s < 1 + \frac{2}{mn}$$

follow easily. We use these convergences to conclude that

$$\begin{aligned} & \int_{\mathbb{R}^n \times [0, \infty)} -\tilde{u} \frac{\partial \varphi}{\partial t} - \tilde{u}^m \Delta \varphi \, dx \, dt \\ &= \lim_{i \rightarrow \infty} \left( \int_{\mathbb{R}^n \times [0, \infty)} -u_i \frac{\partial \varphi}{\partial t} - u_i^{m_i} \Delta \varphi \, dx \, dt \right) = \int_{\mathbb{R}^n} \varphi(x, 0) \, d\mu. \end{aligned}$$

for any test function  $\varphi \in C_0^\infty(\mathbb{R}^n \times [0, \infty))$ . The uniqueness of solutions to the Cauchy problem (5.4) now implies that  $\tilde{u} = u$ . The convergences for the original sequence then follow from the fact that all convergent subsequences converge to the same limit, by uniqueness.  $\square$

*Remark 5.4.* Generalizing Theorem 5.2 to Cauchy problems with growing initial data, described in e.g. [6, Chapters 2 and 3] or [27, Chapter 13], offers no additional difficulties. Indeed, counterparts for all the estimates in Theorem 5.3 above are available; see for instance [27, Theorem 13.1] for appropriate replacements of (5.5) and (5.6). An estimate similar to (5.7) then follows by repeating the above computation. With these estimates in hand, the proof of Theorem 5.2 requires virtually no modifications. We leave the details to the interested reader.

## 6. STABILITY OF DIRICHLET PROBLEMS REVISITED

In this section, we present an alternative proof of Theorem 4.2 in the case  $m_i \geq 1$ . The advantage of this argument is that we avoid using the local  $L^\infty$  estimate, Proposition 4.4.

For the reader's convenience, we give the statement of the theorem before proceeding with the proof, although this is essentially the same as Theorem 4.2 with the additional assumption  $m_i \geq 1$ .

**Theorem 6.1.** *Let  $m_i, i = 1, 2, 3, \dots$ , be a sequence of exponents such that and*

$$m_i \geq 1 \quad \text{and} \quad m_i \rightarrow m \quad \text{as} \quad i \rightarrow \infty.$$

*Let  $u_i, i = 1, 2, 3, \dots$ , be the solutions to*

$$(6.1) \quad \begin{cases} \partial_t u_i - \Delta u_i^{m_i} = 0, & \text{in } \Omega_T, \\ u_i^m = g, & \text{on } \partial\Omega \times [0, T], \\ u_i(x, 0) = u_0 \end{cases}$$

*with fixed initial and boundary values  $g$  and  $u_0$ , where*

$$g \in H^1(0, T; H^1(\Omega)), \quad \text{and} \quad u_0 \in L^{m^++1}(\Omega).$$

*Finally, let  $u$  be the solution to*

$$(6.2) \quad \begin{cases} \partial_t u - \Delta u^m = 0, & \text{in } \Omega_T, \\ u^m = g, & \text{on } \partial\Omega \times [0, T], \\ u(x, 0) = u_0. \end{cases}$$

*with the same boundary and initial values  $g$  and  $u_0$ .*

*Then*

- (1)  $u_i \rightarrow u$  in  $L^q(\Omega_T)$  for all  $1 \leq q < 1 + m$ .
- (2)  $u_i^{m_i} \rightarrow u^m$  in  $L^s(\Omega_T)$  for all  $1 \leq s < 2\kappa$ , where

$$\kappa = 1 + \frac{1}{m} + \frac{1}{mn}.$$

- (3)  $\nabla u_i^{m_i} \rightarrow \nabla u^m$  weakly in  $L^2(\Omega_T)$ .

The following elementary inequality is needed in the proof.

**Lemma 6.2.** *For positive  $t$ , we have*

$$|t^\alpha - t^\beta| \leq c_\varepsilon (1 + t^{\alpha+\varepsilon} + t^{\beta+\varepsilon}) |\alpha - \beta|$$

*Proof.* This follows by an application of the mean value theorem to the function  $x \mapsto t^x$ . We have

$$\frac{d}{dx} t^x = t^x \log t,$$

which is estimated by

$$t^x \log t \leq c_\varepsilon (1 + t^{\alpha+\varepsilon} + t^{\beta+\varepsilon})$$

for  $x$  in the interval  $(\alpha, \beta)$ . □

Stability is a consequence of the following theorem. It provides a quantitative estimate of the difference of two solutions.

**Theorem 6.3.** *Let the exponents  $m_i$  and  $m$ , and the functions  $u_i$  and  $u$  be as in Theorem 6.1. Then*

$$\|u - u_i\|_{L^{1+m}(\Omega_T)} \leq c|m - m_i|^{1/m},$$

for indices  $i$  large enough, where the constant depends on the norms of the boundary and initial values  $g$  and  $u_0$  appearing in Lemma 4.3.

*Proof.* We aim at using the standard inequality

$$(6.3) \quad c|a - b|^{1+m} \leq (a^m - b^m)(a - b)$$

in combination of an application of Oleřnik's test function. The function

$$\eta(x, t) = \begin{cases} \int_t^T u^m - u_i^{m_i} \, ds, & 0 < t < T, \\ 0, & \text{otherwise,} \end{cases}$$

has zero boundary values in Sobolev's sense on the lateral boundary. We test the equations satisfied by  $u$  and  $u_i$ , and subtract the results. This leads to

$$(6.4) \quad \begin{aligned} \int_{\Omega_T} (u - u_i)(u^m - u_i^{m_i}) \, dx \, dt &= - \int_{\Omega_T} \nabla(u^m - u_i^{m_i}) \cdot \int_t^T \nabla(u^m - u_i^{m_i}) \, ds \, dx \, dt \\ &= - \frac{1}{2} \int_{\Omega} \left[ \int_0^T \nabla(u^m - u_i^{m_i}) \, ds \right]^2 \, dx \leq 0, \end{aligned}$$

where we integrated with respect to  $t$  to get the last line. Thus we have

$$\begin{aligned} c \int_{\Omega_T} |u - u_i|^{1+m} \, dx \, dt &\leq \int_{\Omega_T} (u - u_i)(u^m - u_i^m) \, dx \, dt \\ &= \int_{\Omega_T} (u - u_i)(u^m - u_i^{m_i}) \, dx \, dt \\ &\quad + \int_{\Omega_T} (u - u_i)(u_i^{m_i} - u_i^m) \, dx \, dt \\ &\leq \int_{\Omega_T} (u - u_i)(u_i^{m_i} - u_i^m) \, dx \, dt \end{aligned}$$

by (6.3) and (6.4).

We proceed by an application of Hölder's inequality, and get

$$\begin{aligned} \int_{\Omega_T} (u - u_i)(u_i^{m_i} - u_i^m) \, dx \, dt &\leq \left( \int_{\Omega_T} |u - u_i|^{1+m} \, dx \, dt \right)^{1/(1+m)} \\ &\quad \times \left( \int_{\Omega_T} |u_i^{m_i} - u_i^m|^{(m+1)/m} \, dx \, dt \right)^{m/(m+1)}. \end{aligned}$$



Then we apply Lemma 6.2 inside the second integral, and get

$$|u_i^{m_i} - u_i^m| \leq c(1 + u_i^{m_i+\varepsilon} + u_i^{m+\varepsilon})|m - m_i|.$$

The choice of  $\varepsilon$  will be made later. Thus

$$\begin{aligned} & \left( \int_{\Omega_T} |u_i^{m_i} - u_i^m|^{(m+1)/m} dx dt \right)^{m/(m+1)} \\ & \leq c \left( \int_{\Omega_T} (1 + u_i^{m_i+\varepsilon} + u_i^{m+\varepsilon})^{(m+1)/m} dx dt \right)^{m/(m+1)} |m - m_i|. \end{aligned}$$

We put all the estimates together, and end up with

$$\begin{aligned} & \|u - u_i\|_{L^{m+1}(\Omega_T)} \\ & \leq c \left( \int_{\Omega_T} (1 + u_i^{m_i+\varepsilon} + u_i^{m+\varepsilon})^{(m+1)/m} dx dt \right)^{1/(m+1)} |m - m_i|^{1/m} \end{aligned}$$

To finish, we need a uniform (in  $i$ )  $L^1$  bound for the functions

$$(6.5) \quad u_i^{m_i(1+\varepsilon/m_i)(1+1/m)} \quad \text{and} \quad u_i^{m_i(m/m_i+\varepsilon/m_i)(1+1/m)},$$

at least for large  $i$ . We combine the Sobolev embedding (Lemma 2.1) and the energy estimate (Lemma 4.3), and get that  $u_i^{2\kappa_i m_i}$ ,  $i = 1, 2, \dots$ , is bounded in  $L^1(\Omega)$ , where  $\kappa_i$  is given by

$$\kappa_i = 1 + \frac{1}{n} + \frac{1}{m_i n}.$$

The desired conclusion follows by proving that we may choose  $\varepsilon$  small enough, so that the exponents in (6.5) to are less than  $2\kappa_i m_i$  for large  $i$ . Indeed, we may choose  $i$  large enough and  $\varepsilon$  small enough, so that  $1 + \varepsilon/m_i$  and  $m/m_i + \varepsilon/m_i$  are arbitrarily close to one, and so that  $\kappa_i$  is arbitrarily close to  $\kappa$ . Hence to satisfy the conditions

$$m_i(1 + \frac{\varepsilon}{m_i})(1 + \frac{1}{m}) < 2\kappa_i m_i \quad \text{and} \quad m_i(\frac{m}{m_i} + \frac{\varepsilon}{m_i})(1 + \frac{1}{m}) < 2\kappa_i m_i$$

for large  $i$ , it suffices to require that

$$1 + \frac{1}{m} < 2\kappa.$$

A computation shows that this holds if

$$m > \frac{n-2}{n+2},$$

which is guaranteed by our assumption  $m \geq 1$ . Thus above arguments give the desired bound

$$\|u - u_i\|_{L^{1+m}(\Omega_T)} \leq c|m - m_i|^{1/m}$$

for  $i$  large enough.  $\square$

*Remark 6.4.* The reason why the above proof does not work well if  $m < 1$  is the use of (6.3). A similar inequality is of course available for  $m < 1$ , but the exponents that come up in the course of the proof blow up as  $m \rightarrow 1$ .

*Proof of Theorem 6.1.* Theorem 6.3 implies that the sequence  $(u_i)$  converges in  $L^{1+m}(\Omega_T)$  to  $u$ , the solution of the limit problem. For a subsequence we get pointwise a.e. convergence, and an application of Lemma 2.5 then yields the pointwise convergence  $u_i^{m_i} \rightarrow u^m$ . After the pointwise convergences are available, the rest of the convergence claims are established as in the proof of Theorem 4.2.  $\square$

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